J. Math. Anal. Appl. 355 (2009) 231–242

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Positive solutions of a competition model for two resources in the unstirred chemostat ☆

Hua Nie, Jianhua Wu *

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, Shaanxi 710062, People's Republic of China

ARTICLE INFO

Article history:
Received 25 January 2008
Available online 11 February 2009
Submitted by G.F. Webb

Keywords:
Chemostat
Monotone method
Fixed point theory
Multiplicity

ABSTRACT

This paper deals with a competition model between two species for two growth-limiting and perfectly complementary resources in the unstirred chemostat. The main purpose is to determine the exact range of the parameters of two species so that the system possesses positive solutions, and to investigate multiple positive steady states of the system. The main tools used here include the monotone methods and the topological fixed point theory developed by Amann.

1. Introduction

The simplest form of resource-based competition occurs in laboratory apparatus, called a chemostat or continuous culture, which has played an important role in ecology. It can be applied to increase our understanding of both environmental and industrial biotechnological processes. See the monograph of Smith and Waltman [18] for a description of chemostat in detail and for the general theory of the chemostat. Rigorous mathematical analysis of the chemostat model involving one single limiting resource can be found, for example, in [6,7,9–11,14–16,19,21].

In order to identify the growth-limiting resource in ecosystems, specific resources are added to samples taken from the environment and then the stimulation in growth rate is measured. Apparently, the best stimulation of growth is commonly obtained when a combination of resources is supplied, rather than a single resource. When more than one resource is growth-limiting, it is necessary to consider how the resources interact to promote growth. Leon and Tumpson [12], and Rapport [17] classify resources as perfectly complementary, perfectly substitutable, or imperfectly substitutable. Perfectly complementary resources are resources of different essential substances which are independently required for growth, such as a carbon source and a nitrogen source for a bacterium. On the other hand, perfectly substitutable resources are alternative sources of an essential substance, and represent interdependent requirement for growth, such as two carbon sources for phosphorous. The intermediate case is called imperfectly substitutable.

In the past decades, the well-stirred chemostat models with two perfectly complementary resources or two perfectly substitutable resources have been studied extensively, see [2,3,5,8,12,13] and references therein. Considering the environment heterogeneity, the authors in [20,22] removed the well-stirred hypothesis and studied the unstirred chemostat models involving two limiting resources. In [22], the authors have shown that the unique positive solution is globally attracting for the model of single-species growth on two perfectly complementary resources with regard to non-trivial non-negative

☆ The work is supported by the Natural Science Foundation of China, the PhD specialized grant of Ministry of Education of China (No. 200807180004).
* Corresponding author.
E-mail addresses: niehua@snnu.edu.cn (H. Nie), jianhuaw@snnu.edu.cn (J. Wu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.01.045

initial values. For the case of two perfectly substitutable resources, the existence of a positive steady-state solution was determined. In [20], a mathematical model of competition between two species for two perfectly complementary resources in the unstirred chemostat was considered. Some sufficient conditions for the existence of a positive steady-state solution were established analytically. However, the exact parameter range where the model has a positive solution is not clear. On the other hand, numerical simulations in [20] reveal that there are regions in parameter space for which multiple positive steady states may occur. But rigorous mathematical analysis about this result has not been established.

In this paper, we also study the unstirred chemostat model of competition between two species for two perfectly complementary resources, which takes the form of the following reaction–diffusion equations (see [20]):

$$
\begin{aligned}
&S_t = d_1 S_{xx} - u g_1(S,R) - \beta v g_2(S,R), \quad 0<x<1,\ t>0,\\
&R_t = d_2 R_{xx} - \alpha u g_1(S,R) - v g_2(S,R), \quad 0<x<1,\ t>0,\\
&u_t = d_3 u_{xx} + u g_1(S,R), \quad 0<x<1,\ t>0,\\
&v_t = d_4 v_{xx} + v g_2(S,R), \quad 0<x<1,\ t>0,
\end{aligned}
\tag{1.1}
$$

with boundary condition

$$
\begin{aligned}
&S_x(0,t) = -1, \qquad S_x(1,t)+\gamma S(1,t)=0, \qquad u_x(0,t)=u_x(1,t)+\gamma u(1,t)=0,\\
&R_x(0,t) = -1, \qquad R_x(1,t)+\gamma R(1,t)=0, \qquad v_x(0,t)=v_x(1,t)+\gamma v(1,t)=0,
\end{aligned}
$$

where $S(x,t), R(x,t)$ denote the nutrient concentrations at time t, and $u(x,t)$ and $v(x,t)$ denote the biomass of each population in the culture vessel. $d_1, d_2, d_3, d_4 > 0$ are the diffusion rates, $\alpha, \beta, \gamma > 0$ are constants. The response functions are denoted by $g_i(S,R) = \min(p_i(S), q_i(R))$, $i=1,2$, where $p_i(S)$ denotes the response function of the ith population when only resource S is limiting and $q_i(R)$ denotes the response function of the ith population when only resource R is limiting. We will consider the case that the Monod model for exploitative competition for one resource is generalized to the two essential resources case, i.e., $p_i(S) = \frac{m_{s_i} S}{K_{s_i}+S}$, $q_i(R) = \frac{m_{r_i} R}{K_{r_i}+R}$, $i=1,2$, where $m_{s_i}, m_{r_i}, K_{s_i}, K_{r_i}$ are positive constants.

The purpose of this paper is to characterize the exact range of the parameters of two species so that the system possesses positive solutions, and to investigate multiple positive steady states of (1.1). Specifically, we consider the coupled system of the equations

$$
\begin{aligned}
&d_1 S_{xx} - u g_1(S,R) - \beta v g_2(S,R) = 0, \quad 0<x<1,\\
&d_2 R_{xx} - \alpha u g_1(S,R) - v g_2(S,R) = 0, \quad 0<x<1,\\
&d_3 u_{xx} + u g_1(S,R) = 0, \quad 0<x<1,\\
&d_4 v_{xx} + v g_2(S,R) = 0, \quad 0<x<1,\\
&S_x(0) = -1, \qquad S_x(1)+\gamma S(1)=0, \qquad u_x(0)=u_x(1)+\gamma u(1)=0,\\
&R_x(0) = -1, \qquad R_x(1)+\gamma R(1)=0, \qquad v_x(0)=v_x(1)+\gamma v(1)=0.
\end{aligned}
\tag{1.2}
$$

Denote $\Phi_1 = d_1 S + d_3 u + \beta d_4 v$, $\Phi_2 = d_2 R + \alpha d_3 u + d_4 v$. Then Φ_i $(i=1,2)$ satisfies

$$
\Phi_{ixx} = 0, \qquad \Phi_{ix}(0) = -d_i, \qquad \Phi_{ix}(1)+\gamma\Phi_i(1) = 0.
$$

It is easy to see that $\Phi_1 = d_1 z(x)$, $\Phi_2 = d_2 z(x)$, where $z(x) = \frac{1+\gamma}{\gamma} - x$. Hence $S = z - \frac{d_3}{d_1}u - \beta\frac{d_4}{d_1}v$, $R = z - \alpha\frac{d_3}{d_2}u - \frac{d_4}{d_2}v$. Let $\bar{u} = \frac{d_3}{d_1 d_2}u$, $\bar{v} = \frac{d_4}{d_1 d_2}v$, $m = 1/d_3$, $n = 1/d_4$. Then the system (1.2) may be written as

$$
\text{(EP)} \qquad
\begin{aligned}
&u_{xx} + m u g_1(z - d_2 u - d_2\beta v, z - d_1\alpha u - d_1 v) = 0, \quad 0<x<1,\\
&v_{xx} + n v g_2(z - d_2 u - d_2\beta v, z - d_1\alpha u - d_1 v) = 0, \quad 0<x<1,\\
&u_x(0) = u_x(1)+\gamma u(1) = 0, \qquad v_x(0) = v_x(1)+\gamma v(1) = 0.
\end{aligned}
$$

For simplicity, we drop the bars over the non-dimensional quantities.

We are mainly interested in positive solutions of (EP). Hence, there is no loss of generality if we redefine the response functions as follows:

$$
p_i(S) = \begin{cases} \frac{m_{s_i} S}{K_{s_i}+S}, & S \geqslant 0,\\ 0, & S<0, \end{cases} \qquad
q_i(R) = \begin{cases} \frac{m_{r_i} R}{K_{r_i}+R}, & R \geqslant 0,\\ 0, & R<0. \end{cases}
$$

As mentioned before, the main goal of this paper is to determine the exact range of the parameters so that the system possesses positive solutions, and to determine when the numerical simulations results in [20] hold rigorously. The contents of the paper are as follows: In Section 2, we present some basic results and calculate the index of the operator F at neighborhoods of the trivial and semi-trivial non-negative solutions. In Section 3, the boundary of the existence region Σ of positive solutions to (EP) is constructed by two monotone non-decreasing functions $H_1(n), H_2(m)$, and multiplicity of positive steady-state solution is established in certain subregion of Σ.

2. Preliminaries

In this section, we first present some basic results which will be used in this paper. Secondly, we construct an operator with some monotonity and calculate the index of the operator at neighborhoods of the trivial and semi-trivial non-negative fixed points.

Lemma 2.1. *(See [4].) Suppose $q(x) \in C(\overline{\Omega})$, $q(x) > 0$ $(\forall x \in \overline{\Omega})$ and $\gamma(x) \in C(\partial\Omega)$, $\gamma(x) \geqslant 0$ $(\forall x \in \partial\Omega)$. Then all eigenvalues of the problem $\triangle\phi + \lambda q(x)\phi = 0$, $x \in \Omega$, $\frac{\partial\phi}{\partial n} + \gamma(x)\phi = 0$, $x \in \partial\Omega$ can be listed in order $0 < \lambda_1(q(x)) < \lambda_2(q(x)) \leqslant \cdots \to \infty$ with the corresponding eigenfunctions $\phi_1, \phi_2, \ldots,$ where $\phi_1 > 0$ on $\overline{\Omega}$, and the principal eigenvalue $\lambda_1(q) = \inf_\phi \frac{\int_\Omega |\nabla\phi|^2\,dx + \int_{\partial\Omega} \gamma(x)\phi^2\,ds}{\int_\Omega q(x)\phi^2\,dx}$ is simple. Moreover, the comparison principle holds: $\lambda_j(q_1) \leqslant \lambda_j(q_2)$ for $j \geqslant 1$ if $q_1 \geqslant q_2$ on $\overline{\Omega}$ and strict inequality holds if $q_1(x) \not\equiv q_2(x)$.*

Next, we derive some a priori estimates for positive solutions of (EP). To this end, we introduce some notations and recall some well-known facts. Let λ_1, σ_1 be respectively the principal eigenvalues of the problems:

$$\varphi_{1xx} + \lambda_1 g_1(z,z)\varphi_1 = 0 \quad \text{in } (0,1), \qquad \varphi_{1x}(0) = \varphi_{1x}(1) + \gamma\varphi_1(1) = 0;$$

$$\psi_{1xx} + \sigma_1 g_2(z,z)\psi_1 = 0 \quad \text{in } (0,1), \qquad \psi_{1x}(0) = \psi_{1x}(1) + \gamma\psi_1(1) = 0,$$

with the corresponding positive eigenfunctions uniquely determined by the normalization $\max_{[0,1]}\varphi_1 = \max_{[0,1]}\psi_1 = 1$. It is well known (see [20]) that if $m \leqslant \lambda_1$, zero is the unique non-negative solution of the boundary value problem

$$u_{xx} + mug_1(z - d_2u, z - d_1\alpha u) = 0, \quad x \in (0,1), \qquad u_x(0) = u_x(1) + \gamma u(1) = 0, \tag{2.1}$$

and if $m > \lambda_1$, then (2.1) has a unique positive solution, which is denoted by θ_m. Moreover, it satisfies: $0 < \theta_m < \min\{\frac{1}{d_2}, \frac{1}{d_1\alpha}\}z$ on $[0,1]$. Furthermore, one can argue in the exactly similar way as in Lemmas 4.6 and 4.7 of [19] to conclude that $\lim_{m\to\lambda_1}\theta_m = 0$ uniformly for $x \in (0,1)$, $\lim_{m\to\infty}\theta_m = \min\{\frac{1}{d_2}, \frac{1}{d_1\alpha}\}z$ almost everywhere in $(0,1)$.

Similarly, for the other steady-state one-species problem

$$v_{xx} + nvg_2(z - d_2\beta v, z - d_1v) = 0, \quad x \in (0,1), \qquad v_x(0) = v_x(1) + \gamma v(1) = 0, \tag{2.2}$$

the same outcomes hold. For the sake of convenience, we denote the unique positive solution by ϑ_n, which satisfies $0 < \vartheta_n < \min\{\frac{1}{d_2\beta}, \frac{1}{d_1}\}z$.

By similar arguments as in Lemma 2 of [20], we have

Lemma 2.2. *Suppose (u, v) is the non-negative solution of (EP). Then*

(i) $u > 0$ *or* $u \equiv 0$, *and* $v > 0$ *or* $v \equiv 0$;
(ii) $u + \beta v < \frac{z}{d_2}$, $\alpha u + v < \frac{z}{d_1}$;
(iii) $u \leqslant \theta_m$, $v \leqslant \vartheta_n$.

Moreover, $u < \theta_m$ or $u \equiv \theta_m$, and $v < \vartheta_n$ or $v \equiv \vartheta_n$.

Introduce $\hat{\lambda}_1(n), \hat{\sigma}_1(m)$ as the principal eigenvalues of

$$\hat{\varphi}_{1xx} + \hat{\lambda}_1 g_1(z - d_2\beta\vartheta_n, z - d_1\vartheta_n)\hat{\varphi}_1 = 0 \quad \text{in } (0,1), \qquad \hat{\varphi}_{1x}(0) = \hat{\varphi}_{1x}(1) + \gamma\hat{\varphi}_1(1) = 0,$$

$$\hat{\psi}_{1xx} + \hat{\sigma}_1 g_2(z - d_2\theta_m, z - d_1\alpha\theta_m)\hat{\psi}_1 = 0 \quad \text{in } (0,1), \qquad \hat{\psi}_{1x}(0) = \hat{\psi}_{1x}(1) + \gamma\hat{\psi}_1(1) = 0,$$

with the corresponding eigenfunction $\hat{\varphi}_1, \hat{\psi}_1$ normalized by $\max_{[0,1]}\hat{\varphi}_1 = \max_{[0,1]}\hat{\psi}_1 = 1$. It is easy to see that $\hat{\lambda}_1(n) \to \infty$ as $n \to \infty$, $\hat{\sigma}_1(m) \to \infty$ as $m \to \infty$.

Now, repeating the exactly similar arguments as in Theorems 1 and 3 of [20], we can find out the following existence results for positive solutions.

Theorem 2.3. *Suppose $m > \lambda_1$, $n > \sigma_1$ and $(m - \hat{\lambda}_1(n))(n - \hat{\sigma}_1(m)) > 0$. Then there exists a positive steady-state solution (u, v) of (EP) satisfying $0 < u(x) < \theta_m(x)$, $0 < v(x) < \vartheta_n(x)$ for $x \in [0,1]$.*

In [20], the proof of Theorem 2.3 is based on the topological fixed point theory developed by Amann [1]. Thus a compact operator T was introduced in [20]. It is easy to see that the operator T is non-monotonic. However, the main idea of characterizing the existence region of positive solutions to (EP) is based on the monotone method. Thus we need to construct an operator with some monotonity. For this purpose, we introduce the spaces:

$$
\begin{aligned}
&C_B\big([0,1]\big) = \big\{u(x) \in C\big([0,1]\big)\colon\ u_x(0) = u_x(1) + \gamma u(1) = 0\big\},\\
&X = C\big([0,1]\big) \times C\big([0,1]\big),\\
&W = \big\{(u,v) \in X\colon\ u \geqslant 0,\ \ v \geqslant 0 \text{ for } x \in [0,1]\big\},\\
&D = \big\{(u,v) \in W\colon\ \|u\| + \|v\| \leqslant R_0,\ \ \|\cdot\| \text{ is the usual norm in } C([0,1])\big\},
\end{aligned}
$$

where $R_0 = 2\max\{\frac{1}{d_1}, \frac{1}{d_2}, \frac{1}{d_1\alpha}, \frac{1}{d_2\beta}\}\|z\|$. Define $F : X \to X$ as

$$
F(u,v) := \left(-\frac{d^2}{dx^2} + M\right)^{-1} \begin{pmatrix} mug_1(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) + Mu \\ nvg_2(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) + Mv \end{pmatrix},
$$

where $(-\frac{d^2}{dx^2} + M)^{-1}$ is the inverse operator of $-\frac{d^2}{dx^2} + M$ subject to the boundary conditions $u_x(0) = u_x(1) + \gamma u(1) = 0$, and M is large enough such that $M + mg_1(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) > 0$ and $M + ng_2(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) > 0$ for all $(u,v) \in D$. Clearly, F is compact. Moreover, $F : D \to W$ is a continuously differentiable monotonic operator with respect to the cone $P = \{(u,v) \in X\colon\ u \geqslant 0,\ \ v \leqslant 0 \text{ for } x \in [0,1]\}$. It follows from Lemma 2.2 that (EP) has non-negative solutions if and only if the operator F has a fixed point in D. Obviously, all of the trivial and semi-trivial non-negative fixed points of F include $(0,0)$, $(\theta_m, 0)$, $(0, \vartheta_n)$. In order to use the degree theory, we need to calculate the index of the operator F at neighborhoods of these non-negative fixed points.

Lemma 2.4. *For $\lambda \geqslant 1$, the equation $F(u,v) = \lambda(u,v)$ has no solution in W satisfying $\|u\| + \|v\| = R_0$.*

Proof. Suppose $(u,v) \in W$ satisfies $F(u,v) = \lambda(u,v)$ and $\|u\| + \|v\| = R_0$. Then

$$
\begin{aligned}
&u_{xx} + \lambda^{-1} mug_1(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) = \frac{\lambda - 1}{\lambda} Mu,\\
&v_{xx} + \lambda^{-1} nvg_2(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) = \frac{\lambda - 1}{\lambda} Mv,\\
&u_x(0) = u_x(1) + \gamma u(1) = 0, \qquad v_x(0) = v_x(1) + \gamma v(1) = 0.
\end{aligned}
$$

Define $w = u + \beta v - z/d_2$. Then w satisfies

$$
\begin{aligned}
&w_{xx} + \lambda^{-1} mug_1(-d_2 w, z - d_1\alpha u - d_1 v) + \lambda^{-1} n\beta vg_2(-d_2 w, z - d_1\alpha u - d_1 v) = \frac{\lambda - 1}{\lambda} M(u + \beta v),\\
&w_x(0) = \frac{1}{d_2}, \qquad w_x(1) + \gamma w(1) = 0.
\end{aligned}
\tag{2.3}
$$

First we show that $w \leqslant 0$ on $[0,1]$ by contradiction. Assume that $w(1) > 0$. It follows from the boundary conditions that $w_x(1) < 0$. Hence, there exists $x_0 \in [0,1)$ such that for all $x \in (x_0, 1]$, $w(x) > 0$, and either $x_0 = 0$ or $w(x_0) = 0$. From Eq. (2.3), one can claim that for all $x \in [x_0, 1]$, $w_{xx} \geqslant 0$. Thus $w_x(x) \leqslant w_x(1) < 0$. That is, $w(x)$ is decreasing on $[x_0, 1]$. Since $w_x(0) = 1/d_2 > 0$, we know that $x_0 \neq 0$. Namely, $x_0 > 0$ and $w(x_0) = 0$, which contradicts that $w(x)$ is decreasing on $[x_0, 1]$ and $w(1) > 0$. Therefore, $w(1) \leqslant 0$. Next, assume there exists $\bar{x} \in [0,1)$ with $w(\bar{x}) > 0$. Then there exist $\delta_1 \geqslant 0$ and $\delta_2 > 0$ such that $w(x) > 0$ for all $x \in (\bar{x} - \delta_1, \bar{x} + \delta_2) \subset (0,1)$, $w(\bar{x} + \delta_2) = 0$ and either $\bar{x} - \delta_1 = 0$ or $w(\bar{x} - \delta_1) = 0$. Similarly, for all $x \in (\bar{x} - \delta_1, \bar{x} + \delta_2)$, $w_{xx}(x) \geqslant 0$, and hence $w_x(x) \leqslant w_x(\bar{x} + \delta_2)$. Since $w(\bar{x} + \delta_2) = 0$, it follows that $w_x(\bar{x} + \delta_2) \leqslant 0$, and so $w(x)$ is non-increasing on $[\bar{x} - \delta_1, \bar{x} + \delta_2]$. Then $\bar{x} - \delta_1 \neq 0$ based on $w_x(0) = 1/d_2$, and so $w(\bar{x} - \delta_1) = 0$. Therefore, $w(x) \equiv 0$ on $[\bar{x} - \delta_1, \bar{x} + \delta_2]$, a contradiction. Hence, $u + \beta v \leqslant z/d_2$ on $[0,1]$, which implies that $u + v \leqslant \max\{\frac{1}{d_2}, \frac{1}{d_2\beta}\}z$.

On the other hand, by similar arguments, $\alpha u + v \leqslant z/d_1$ on $[0,1]$, which implies that $u + v \leqslant \max\{\frac{1}{d_1}, \frac{1}{d_1\alpha}\}z$. That is, $u + v \leqslant \max\{\frac{1}{d_1}, \frac{1}{d_2}, \frac{1}{d_1\alpha}, \frac{1}{d_2\beta}\}\|z\| = R_0/2$. Hence, there exists no solution of $F(u,v) = \lambda(u,v)$ in W satisfying $\|u\| + \|v\| = R_0$. $\square$

As a consequence of Lemma 2.4 and Lemma 12.1 in [1], we have the following outcome.

Lemma 2.5. $\operatorname{index}(F, \dot{D}, W) = 1$, *where $\dot{D}$ denotes the interior of D in W.*

Lemma 2.6. *Suppose $m > \lambda_1$, $n > \sigma_1$. Then for $\delta > 0$ small enough,* $\operatorname{index}(F, P_\delta(0,0), W) = 0$, *where $P_\delta(0,0) = \{(u,v) \in W\colon\ \|u\| + \|v\| < \delta\}$.*

Proof. Noting that the definition of θ_m, ϑ_n, for given $\epsilon_0 > 0$ sufficiently small, we can take $0 < \delta < \delta_0 \ll 1$ such that $\frac{\delta}{\gamma} \leqslant \min\{\theta_m - \epsilon_0, \vartheta_n - \epsilon_0\}$. Denote $S_\delta^+ = \{(u, v) \in W\colon \|u\| + \|v\| = \frac{\delta}{\gamma}\}$. Thus $\|u\| \leqslant \delta z$, $\|v\| \leqslant \delta z$ whenever $(u, v) \in S_\delta^+$.

Let $\psi = 2 + \gamma - \gamma x^2$. Then $\psi > 0$ on $[0, 1]$ and satisfies

$$\psi_{xx} < 0 \quad \text{in } (0, 1), \qquad \psi_x(0) = 0, \qquad \psi_x(1) + \gamma\psi(1) = 0.$$

Moreover, $\Psi = (\psi, \psi) \in W$. Next, we show that for $\lambda \geqslant 0$, $(u, v) - F(u, v) = \lambda(\psi, \psi)$ has no solution on S_δ^+ for small δ. Assume on the contrary that this problem has a solution (u, v) on S_δ^+. By the definition of ψ, one can find that (u, v) satisfies

$$\begin{aligned}
&u_{xx} + mug_1(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) = \lambda(\psi_{xx} - M\psi) \leqslant 0, \quad 0 < x < 1,\\
&v_{xx} + nvg_2(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1 v) = \lambda(\psi_{xx} - M\psi) \leqslant 0, \quad 0 < x < 1,\\
&u_x(0) = u_x(1) + \gamma u(1) = 0, \qquad v_x(0) = v_x(1) + \gamma v(1) = 0.
\end{aligned}$$

By virtue of $m > \lambda_1$, $n > \sigma_1$, we can take $\delta_1 \ll 1$ such that for any $0 < \delta < \delta_1$, $m > \lambda_1(g_1((1 - \delta d_2\beta)z, (1 - \delta d_1)z))$ and $n > \lambda_1(g_2((1 - \delta d_2)z, (1 - \delta d_1\alpha)z))$, where $\lambda_1(q(x))$ is given by Lemma 2.1. Hence, by similar arguments as in Lemma 3.2 of [19], we can show that the following two problems have unique positive solution u^*, v^*, respectively,

$$\begin{aligned}
&u^*_{xx} + mu^*g_1\big((1 - \delta d_2\beta)z - d_2u^*, (1 - \delta d_1)z - d_1\alpha u^*\big) = 0, \quad 0 < x < 1,\\
&v^*_{xx} + nv^*g_2\big((1 - \delta d_2)z - d_2\beta v^*, (1 - \delta d_1\alpha)z - d_1 v^*\big) = 0, \quad 0 < x < 1,
\end{aligned}$$

with the usual boundary conditions. It follows from the monotone method and the uniqueness of u^*, v^* that $u \geqslant u^*$, $v \geqslant v^*$. On the other hand, by L^p estimate and the Sobolev embedding theorem, we proceed as in the proof of Theorem 2.5 in [22] to obtain

$$\lim_{\delta\to 0} u^* = \theta_m, \qquad \lim_{\delta\to 0} v^* = \vartheta_n.$$

Hence, there exists $\delta_2 > 0$ such that for $\delta < \delta_2$, $u^* > \theta_m - \epsilon_0$, $v^* > \vartheta_n - \epsilon_0$. Set $\bar\delta = \min\{\delta_0, \delta_1, \delta_2\}$. Then for any $\delta < \bar\delta$, we can find that $u > \frac{\delta}{\gamma}$, $v > \frac{\delta}{\gamma}$, which contradicts $(u, v) \in S_\delta^+$. Hence $\operatorname{index}(F, P_\delta(0, 0), W) = 0$ by Lemma 12.1 of [1]. □

Let $O^+(\theta_m, 0)$ and $O^+(0, \vartheta_n)$ be a small neighborhood of $(\theta_m, 0)$ and $(0, \vartheta_n)$ in W, respectively. Next, we calculate the index of the operator F at $O^+(\theta_m, 0)$ and $O^+(0, \vartheta_n)$.

Lemma 2.7. *Suppose that $m > \lambda_1$, $n > \sigma_1$ and F has no fixed point in $\dot D$. Then*

(i) $\operatorname{index}(F, O^+(\theta_m, 0), W) = 1$ *if* $n < \hat\sigma_1(m)$ *and* $\operatorname{index}(F, O^+(\theta_m, 0), W) = 0$ *if* $n > \hat\sigma_1(m)$;
(ii) $\operatorname{index}(F, O^+(0, \vartheta_n), W) = 1$ *if* $m < \hat\lambda_1(n)$ *and* $\operatorname{index}(F, O^+(0, \vartheta_n), W) = 0$ *if* $m > \hat\lambda_1(n)$.

Proof. Here we only prove (i), because the proof of (ii) is similar. For this purpose, we define

$$F(t)(u, v) = \left(-\frac{d^2}{dx^2} + M\right)^{-1} \begin{pmatrix} mug_1(z - d_2u - td_2\beta v, z - d_1\alpha u - td_1 v) + Mu \\ nvg_2(z - d_2u - td_2\beta v, z - d_1\alpha u - td_1 v) + Mv \end{pmatrix}.$$

Then $F(t)(u, v) = (u, v)$ leads to

$$\begin{aligned}
&u_{xx} + mug_1(z - d_2u - td_2\beta v, z - d_1\alpha u - td_1 v) = 0,\\
&v_{xx} + nvg_2(z - d_2u - td_2\beta v, z - d_1\alpha u - td_1 v) = 0.
\end{aligned} \tag{2.4}$$

If (u, v) is a fixed point of $F(t)$ on the boundary $\partial O^+(\theta_m, 0)$ of $O^+(\theta_m, 0)$ in W, it is easy to see that $u > 0$, $v \geqslant 0$. Furthermore, we can show that $v > 0$, otherwise we have $(u, v) = (\theta_m, 0)$. This is a contradiction to $(u, v) \in \partial O^+(\theta_m, 0)$.

Next, we show that for $t \in [0, 1]$, $F(t)$ has no fixed point on $\partial O^+(\theta_m, 0)$. Assume on the contrary that $(u, v) \in \partial O^+(\theta_m, 0)$ is a fixed point of $F(t)$. Then $u > 0$, $v > 0$ by the above arguments. But for $t = 0$, we can find that $u = \theta_m$, $v = 0$ based on $m > \lambda_1$, $n \neq \hat\sigma_1(m)$, a contradiction. For $t > 0$, Eq. (2.4) indicates $(u, tv) > (0, 0)$ is a fixed point of F in $\dot D$, which is a contradiction to the hypothesis of this lemma. Thus by the homotopy invariance of topological degree that

$$\operatorname{index}\big(F, O^+(\theta_m, 0), W\big) = \operatorname{index}\big(F(1), O^+(\theta_m, 0), W\big) = \operatorname{index}\big(F(0), O^+(\theta_m, 0), W\big),$$

where $F(0)(u, v) = (-\frac{d^2}{dx^2} + M)^{-1}(mug_1(z - d_2u, z - d_1\alpha u) + Mu, nvg_2(z - d_2u, z - d_1\alpha u) + Mv)$.

The remain task is to calculate $\operatorname{index}(F(0), O^+(\theta_m, 0), W)$. For this purpose, we first find out the fixed point (u, v) of $F(0)$. Suppose (u, v) is a fixed point of $F(0)$ in $O^+(\theta_m, 0)$. Then $u > 0$, $v \geqslant 0$ and

$$u_{xx}+mug_1(z-d_2u,z-d_1\alpha u)=0, \qquad u_x(0)=u_x(1)+\gamma u(1)=0,$$
$$v_{xx}+nvg_2(z-d_2u,z-d_1\alpha u)=0, \qquad v_x(0)=v_x(1)+\gamma v(1)=0. \tag{2.5}$$

It is easy to find that $u=\theta_m$ based on $m>\lambda_1$. On the other hand, if $n<\hat{\sigma}_1(m)$ (or $n>\hat{\sigma}_1(m)$), one can claim that the principal eigenvalue $\bar{\sigma}_1(m)$ of the following problem is negative (or positive)

$$\bar{\phi}_{xx}+ng_2(z-d_2\theta_m,z-d_1\alpha\theta_m)\bar{\phi}=\bar{\sigma}_1(m)\bar{\phi}, \qquad \bar{\phi}_x(0)=\bar{\phi}_x(1)+\gamma\bar{\phi}(1)=0.$$

Hence, $v\equiv 0$ by substituting $u=\theta_m$ into the second equation of (2.5). Thus, $(\theta_m,0)$ is the unique fixed point of $F(0)$ in $O^+(\theta_m,0)$, and

$$\operatorname{index}\big(F(0),O^+(\theta_m,0),W\big)=\operatorname{index}\big(F(0),(\theta_m,0),W\big).$$

For $\tau\in[0,1]$, let $T(\tau)$ be defined by

$$T(\tau)(u,v)=\left(-\frac{d^2}{dx^2}+M\right)^{-1}\begin{pmatrix} mug_1(z-d_2u,z-d_1\alpha u)+Mu \\ nvg_2(z-d_2(\tau\theta_m+(1-\tau)u),z-d_1\alpha(\tau\theta_m+(1-\tau)u))+Mv\end{pmatrix}.$$

Then $T(\tau)(u,v)=(u,v)$ satisfies

$$u_{xx}+mug_1(z-d_2u,z-d_1\alpha u)=0,$$
$$v_{xx}+nvg_2\big(z-d_2\big(\tau\theta_m+(1-\tau)u\big),z-d_1\alpha\big(\tau\theta_m+(1-\tau)u\big)\big)=0,$$
$$u_x(0)=u_x(1)+\gamma u(1)=0, \qquad v_x(0)=v_x(1)+\gamma v(1)=0. \tag{2.6}$$

Next, we show that $T(\tau)$ has no fixed point on $\partial O^+(\theta_m,0)\cap W$. Otherwise, it follows from the first equation of (2.6) that $u=\theta_m$, and substituting this into the second equation of (2.6), we obtain that $v\equiv 0$ based on $n\neq\hat{\sigma}_1(m)$. Hence the only fixed point of $T(\tau)$ on $\partial O^+(\theta_m,0)$ is $(\theta_m,0)$, a contradiction. On the other hand, it is easy to see that

$$F(0)=T(0), \qquad T(1)=T_1\times T_2,$$

where $T_1u=(-\frac{d^2}{dx^2}+M)^{-1}(mug_1(z-d_2u,z-d_1\alpha u)+Mu)$, $T_2v=(-\frac{d^2}{dx^2}+M)^{-1}(nvg_2(z-d_2\theta_m,z-d_1\alpha\theta_m)+Mv)$, $(T_1\times T_2)(u,v)=(T_1u,T_2v)$. Hence, by the homotopy invariance of topological degree and the product theorem for fixed points that

$$\operatorname{index}\big(F(0),(\theta_m,0),W\big)=\operatorname{index}\big(T(0),(\theta_m,0),W\big)=\operatorname{index}\big(T(1),(\theta_m,0),W\big)=\operatorname{index}(T_1,\theta_m,C_B)\cdot\operatorname{index}\big(T_2,0,C_B^+\big).$$

Noting that T_2 is a linear compact operator, one can claim that T_2 has no eigenvalue greater than 1 with positive eigenfunction in C_B^+ provided $n<\hat{\sigma}_1(m)$; and T_2 possesses an eigenvalue greater than 1 with positive eigenfunction in C_B^+ provided $n>\hat{\sigma}_1(m)$. Thus it follows from Lemma 13.1 of [1] that $\operatorname{index}(T_2,0,C_B^+)=1$ provided $n<\hat{\sigma}_1(m)$, and $\operatorname{index}(T_2,0,C_B^+)=0$ provided $n>\hat{\sigma}_1(m)$.

Next, we show that $\operatorname{index}(T_1,\theta_m,C_B)=1$. Let $\delta=2\min\{\frac{1}{d_2},\frac{1}{d_1\alpha}\}\|z\|$, $P_\delta=\{u\in C_B^+\colon \|u\|\leqslant\delta\}$, $\partial P_\delta=\{u\in C_B^+\colon \|u\|=\delta\}$. For $\lambda\geqslant 1$, $T_1u=\lambda u$ leads to

$$u_{xx}+\lambda^{-1}mug_1(z-d_2u,z-d_1\alpha u)=\frac{\lambda-1}{\lambda}Mu, \qquad u_x(0)=u_x(1)+\gamma u(1)=0.$$

By the similar arguments as in the proof of Lemma 2.2, we can show that $u\leqslant\min\{\frac{1}{d_2},\frac{1}{d_1\alpha}\}z<\delta$. Hence for $\lambda\geqslant 1$, $T_1u=\lambda u$ has no solution on ∂P_δ. It follows from Lemma 12.1 of [1] that $\operatorname{index}(T_1,P_\delta,C_B^+)=1$. Let $0<\delta_0\leqslant\frac{1}{2}\min_{[0,1]}\{\theta_m\}$. Suppose that for $\lambda\geqslant 0$, $p=2+\gamma-\gamma x^2$, the equation $u-T_1u=\lambda p$ has a solution u on ∂P_{δ_0}. Then we have

$$u_{xx}+mug_1(z-d_2u,z-d_1\alpha u)=\lambda p_{xx}-\lambda Mp\leqslant 0,$$

which implies that u is a super-solution of (2.1). By the monotone method and the uniqueness of θ_m, one can assert that $u\geqslant\theta_m$. This is a contradiction to $\|u\|=\delta_0$. Hence, $\operatorname{index}(T_1,P_{\delta_0},C_B^+)=0$. Since $u=\theta_m$ is the unique fixed point of T_1 in $P_\delta\setminus P_{\delta_0}$, we have

$$\operatorname{index}(T_1,\theta_m,C_B)=\operatorname{index}\big(T_1,P_\delta\setminus P_{\delta_0},C_B^+\big)=\operatorname{index}\big(T_1,P_\delta,C_B^+\big)-\operatorname{index}\big(T_1,P_{\delta_0},C_B^+\big)=1.$$

Combining the above results, we obtain

$$\begin{aligned}\operatorname{index}\big(F,O^+(\theta_m,0),W\big)&=\operatorname{index}\big(F(0),O^+(\theta_m,0),W\big)=\operatorname{index}\big(F(0),(\theta_m,0),W\big)\\&=\operatorname{index}\big(T(1),(\theta_m,0),W\big)=\operatorname{index}(T_1,\theta_m,C_B)\cdot\operatorname{index}\big(T_2,0,C_B^+\big)\\&=\begin{cases}1 & \text{provided } n<\hat{\sigma}_1(m),\\ 0 & \text{provided } n>\hat{\sigma}_1(m).\end{cases}\qquad\square\end{aligned}$$

Remark. According to Lemmas 2.5–2.7, we can also assert that Theorem 2.3 holds.

3. Characterization of the existence region for positive solutions

The goal of this section is to show that the existence region Σ of positive solutions to (EP) is convex, and its boundary consists of two curves which are determined by two monotone non-decreasing functions $G_1(m,n)$ and $G_2(m,n)$. For this purpose, we first construct the functions G_1, G_2.

Our approach to this problem is by the method of super- and sub-solutions. Note that $g_1(z-d_2u-d_2\beta v, z-d_1\alpha u-d_1 v)$ is monotone decreasing in v and $g_2(z-d_2u-d_2\beta v, z-d_1\alpha u-d_1 v)$ is monotone decreasing in u. The pairs $(\tilde u, \tilde v)$ and $(\hat u, \hat v)$ are called super- and sub-solutions of (EP) if

$$\begin{aligned}
&\tilde u_{xx}+m\tilde u g_1(z-d_2\tilde u-d_2\beta\hat v, z-d_1\alpha\tilde u-d_1\hat v)\leqslant 0, \quad 0<x<1,\\
&\tilde v_{xx}+n\tilde v g_2(z-d_2\hat u-d_2\beta\tilde v, z-d_1\alpha\hat u-d_1\tilde v)\leqslant 0, \quad 0<x<1,\\
&\hat u_{xx}+m\hat u g_1(z-d_2\hat u-d_2\beta\tilde v, z-d_1\alpha\hat u-d_1\tilde v)\geqslant 0, \quad 0<x<1,\\
&\hat v_{xx}+n\hat v g_2(z-d_2\tilde u-d_2\beta\hat v, z-d_1\alpha\tilde u-d_1\hat v)\geqslant 0, \quad 0<x<1,\\
&\tilde u_x(0)\geqslant 0\geqslant \hat u_x(0), \qquad \tilde u_x(1)+\gamma\tilde u(1)\geqslant 0\geqslant \hat u_x(1)+\gamma\hat u(1),\\
&\tilde v_x(0)\geqslant 0\geqslant \hat v_x(0), \qquad \tilde v_x(1)+\gamma\tilde v(1)\geqslant 0\geqslant \hat v_x(1)+\gamma\hat v(1).
\end{aligned}$$

Furthermore, if $\hat u\leqslant\tilde u$ and $\hat v\leqslant\tilde v$, then the pairs $(\tilde u,\tilde v)$ and $(\hat u,\hat v)$ are called ordered super- and sub-solutions of (EP).

Let $\underline{u}_0, \underline{v}_0$ be the maximal non-negative solutions of the following problems, respectively,

$$u_{xx}+mug_1(z-d_2u-d_2\beta\vartheta_n, z-d_1\alpha u-d_1\vartheta_n)=0, \qquad u_x(0)=u_x(1)+\gamma u(1)=0; \tag{3.1}$$

$$v_{xx}+nvg_2(z-d_2\theta_m-d_2\beta v, z-d_1\alpha\theta_m-d_1 v)=0, \qquad v_x(0)=v_x(1)+\gamma v(1)=0. \tag{3.2}$$

Then it is easy to see that $\underline{u}_0\equiv 0$ if $m\leqslant\hat\lambda_1(n)$; and $\underline{u}_0>0$ is the unique positive solution of (3.1) if $m>\hat\lambda_1(n)$. Moreover, by the monotone method and the uniqueness of positive solution to (2.1), one can assert that $\underline{u}_0\leqslant\theta_m$. Similarly, $\underline{v}_0\equiv 0$ if $n\leqslant\hat\sigma_1(m)$; and $\underline{v}_0>0$ is the unique positive solution of (3.2) if $n>\hat\sigma_1(m)$. Moreover, $\underline{v}_0\leqslant\vartheta_n$.

Next, let $\overline{u}_0, \overline{v}_0$ be the maximal non-negative solutions of the following problems, respectively,

$$u_{xx}+mug_1(z-d_2u-d_2\beta\underline{v}_0, z-d_1\alpha u-d_1\underline{v}_0)=0, \qquad u_x(0)=u_x(1)+\gamma u(1)=0; \tag{3.3}$$

$$v_{xx}+nvg_2(z-d_2\underline{u}_0-d_2\beta v, z-d_1\alpha\underline{u}_0-d_1 v)=0, \qquad v_x(0)=v_x(1)+\gamma v(1)=0. \tag{3.4}$$

By the monotone method and the uniqueness of positive solutions to (2.1) or (2.2), one can claim that

$$0\leqslant\underline{u}_0\leqslant\overline{u}_0\leqslant\theta_m, \qquad 0\leqslant\underline{v}_0\leqslant\overline{v}_0\leqslant\vartheta_n.$$

Moreover, it is easy to verify that the pairs $(\overline{u}_0,\overline{v}_0)$ and $(\underline{u}_0,\underline{v}_0)$ are ordered super- and sub-solutions of (EP). Let $(\overline{u}(x,t),\underline{v}(x,t))$ and $(\underline{u}(x,t),\overline{v}(x,t))$ be the solutions of the time-dependent problem

$$\begin{aligned}
&u_t-u_{xx}=mug_1(z-d_2u-d_2\beta v, z-d_1\alpha u-d_1 v), \quad 0<x<1,\ t>0,\\
&v_t-v_{xx}=nvg_2(z-d_2u-d_2\beta v, z-d_1\alpha u-d_1 v), \quad 0<x<1,\ t>0,\\
&u_x(0,t)=u_x(1,t)+\gamma u(1,t)=0, \qquad v_x(0,t)=v_x(1,t)+\gamma v(1,t)=0, \quad t>0,
\end{aligned} \tag{3.5}$$

with $(\overline{u}(x,0),\underline{v}(x,0))=(\overline{u}_0,\underline{v}_0)$ and $(\underline{u}(x,0),\overline{v}(x,0))=(\underline{u}_0,\overline{v}_0)$, respectively. Noting that the couple system is quasi-monotone non-increasing and the initial conditions $(\overline{u}_0,\underline{v}_0)$ and $(\underline{u}_0,\overline{v}_0)$ are ordered super- and sub-solutions of the steady-state problem of (3.5), we can claim that the time-dependent functions $\overline{u}(x,t)$ and $\overline{v}(x,t)$ are monotone non-increasing in t, and $\underline{u}(x,t), \underline{v}(x,t)$ are monotone non-decreasing in t. Moreover, the limits

$$\lim_{t\to\infty}\big(\overline{u}(x,t),\underline{v}(x,t)\big)=\big(\overline{u}_s(x),\underline{v}_s(x)\big), \qquad \lim_{t\to\infty}\big(\underline{u}(x,t),\overline{v}(x,t)\big)=\big(\underline{u}_s(x),\overline{v}_s(x)\big)$$

exist and are solutions of (EP). Define

$$G_1(m,n)=\lambda_1\big(g_1(z-d_2\beta\underline{v}_s, z-d_1\underline{v}_s)\big), \qquad G_2(m,n)=\lambda_1\big(g_2(z-d_2\underline{u}_s, z-d_1\alpha\underline{u}_s)\big),$$

where $\lambda_1(q(x))$ is given by Lemma 2.1. Next, we show that the set Σ can be described by the set of $\{(m,n)\mid m>G_1(m,n), n>G_2(m,n)\}$.

Theorem 3.1. *Let $m\neq\hat\lambda_1(n)$, $n\neq\hat\sigma_1(m)$. Then* (EP) *has a positive solution* (*i.e.*, $(m,n)\in\Sigma$) *if and only if $m>G_1(m,n)$ and $n>G_2(m,n)$.*

Proof. Let $(m,n) \in \Sigma$ and $(u(x), v(x))$ be a positive solution of (EP). Then it follows from Lemma 2.2 that

$$0 < u(x) \leqslant \theta_m, \qquad 0 < v(x) \leqslant \vartheta_n.$$

Claim $\underline{u}_0 \leqslant u$ and $\underline{v}_0 \leqslant v$. If $m \leqslant \hat{\lambda}_1(n)$, then $\underline{u}_0 \equiv 0 \leqslant u$. If $m > \hat{\lambda}_1(n)$, then $\underline{u}_0 > 0$ is the unique positive solution to (3.1). By the monotone method and the uniqueness of positive solution to (3.1), we can assert that $\underline{u}_0 \leqslant u$ based on $v(x) \leqslant \vartheta_n$. Similarly, we have $\underline{v}_0 \leqslant v$. Repeating the similar arguments as for $\underline{u}_0, \underline{v}_0$, we obtain that $u \leqslant \overline{u}_0$, $v \leqslant \overline{v}_0$. Thus the comparison principle of the parabolic equations leads to

$$\underline{u}(x,t) \leqslant u(x) \leqslant \overline{u}(x,t), \qquad \underline{v}(x,t) \leqslant v(x) \leqslant \overline{v}(x,t) \quad \text{in } (0,1) \times [0,+\infty).$$

By virtue of the monotone convergence property of $(\overline{u}(x,t), \underline{v}(x,t))$ and $(\underline{u}(x,t), \overline{v}(x,t))$ with $(\overline{u}(x,0), \underline{v}(x,0)) = (\overline{u}_0, \underline{v}_0)$ and $(\underline{u}(x,t), \overline{v}(x,t)) = (\underline{u}_0, \overline{v}_0)$, the above inequalities imply that $\underline{u}_s \leqslant u \leqslant \overline{u}_s$, $\underline{v}_s \leqslant v \leqslant \overline{v}_s$. On the other hand, since $(\overline{u}_s, \underline{v}_s)$ and $(\underline{u}_s, \overline{v}_s)$ are solutions of (EP), it follows that $\overline{u}_s$ and $\overline{v}_s$ are the positive solutions of the scalar boundary value problem, respectively,

$$u_{xx} + mug_1(z - d_2u - d_2\beta\underline{v}_s, z - d_1\alpha u - d_1\underline{v}_s) = 0, \qquad u_x(0) = u_x(1) + \gamma u(1) = 0; \tag{3.6}$$

$$v_{xx} + nvg_2(z - d_2\underline{u}_s - d_2\beta v, z - d_1\alpha\underline{u}_s - d_1v) = 0, \qquad v_x(0) = v_x(1) + \gamma v(1) = 0. \tag{3.7}$$

This implies that

$$m > \lambda_1\big(g_1(z - d_2\beta\underline{v}_s, z - d_1\underline{v}_s)\big) \equiv G_1(m,n), \qquad n > \lambda_1\big(g_2(z - d_2\underline{u}_s, z - d_1\alpha\underline{u}_s)\big) \equiv G_2(m,n).$$

Namely, when $(m,n) \in \Sigma$, one must have $m > G_1(m,n)$, $n > G_2(m,n)$.

Conversely, suppose $(m,n) \in \{(m,n) \mid m > G_1(m,n),\ n > G_2(m,n)\}$ and $m \neq \hat{\lambda}_1(n)$, $n \neq \hat{\sigma}_1(m)$. Then there are four possibilities:

(i) $m < \hat{\lambda}_1(n)$, $n < \hat{\sigma}_1(m)$,
(ii) $m > \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$,
(iii) $m < \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$,
(iv) $m > \hat{\lambda}_1(n)$, $n < \hat{\sigma}_1(m)$.

It follows from Theorem 2.3 that for the cases (i) and (ii), (EP) has at least one positive solution, that is, $(m,n) \in \Sigma$. The remain task is to show that $(m,n) \in \Sigma$ for the cases (iii) and (iv). Since the proof for these two cases is similar, we only consider the case (iii): $m < \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$. First, since $n > \hat{\sigma}_1(m)$, it is easy to see that the problem (3.2) has only one positive solution $\underline{v}_0$. Hence the solution $(\overline{u}(x,t), \underline{v}(x,t))$ of (3.5) with $(\overline{u}(x,0), \underline{v}(x,0)) = (\overline{u}_0, \underline{v}_0)$ satisfies

$$\underline{v}_s(x) \geqslant \underline{v}(x,t) \geqslant \underline{v}_0(x) > 0 \quad \text{for all } x \in (0,1),\ t > 0,$$

where $\underline{v}_s(x) = \lim_{t\to\infty} \underline{v}(x,t)$. On the other hand, since $m > G_1(m,n)$, we can find that (3.6) has a unique positive solution, which is denoted by $u(x)$. Noting that $\underline{v}_s \geqslant \underline{v}_0$, a comparison between the solutions of Eqs. (3.3) and (3.6) ensures that $\overline{u}_0(x) \geqslant u(x) > 0$ in $(0,1)$. Thus $\overline{u}_0$ and u are ordered super- and sub-solutions of the following time-dependent problem

$$u_t - u_{xx} = mug_1\big(z - d_2u - d_2\beta\underline{v}(x,t), z - d_1\alpha u - d_1\underline{v}(x,t)\big),$$
$$u_x(0,t) = u_x(1,t) + \gamma u(1,t) = 0, \qquad u(x,0) = \overline{u}_0(x).$$

Since $\overline{u}(x,t)$ is the solution of the above problem, it follows that $\overline{u}(x,t) \geqslant u(x) > 0$ for all $x \in (0,1)$ and $t \geqslant 0$. This implies that

$$\overline{u}_s(x) = \lim_{t\to\infty} \overline{u}(x,t) \geqslant u(x) > 0$$

and $(\overline{u}_s, \underline{v}_s)$ is a positive solution of (EP), that is, $(m,n) \in \Sigma$. The proof is completed. $\square$

Now, we begin to characterize the boundary of Σ. To this end, we first give the following lemma.

Lemma 3.2. *Let $(\tilde{u}, \tilde{v})$, $(\hat{u}, \hat{v})$ be ordered super- and sub-solutions of (EP), and $I = \{(u,v)\colon\ \hat{u} \leqslant u \leqslant \tilde{u},\ \hat{v} \leqslant v \leqslant \tilde{v}\}$. Then $FI \subset I$.*

Proof. In view of the smooth and bounded property of g_1, g_2, we know that there exists $M > 0$ sufficiently large such that $F_1(u,v)$ is monotone non-decreasing in u and $F_2(u,v)$ is monotone non-decreasing in v, where

$$F_1(u,v) = mug_1(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1v) + Mu,$$
$$F_2(u,v) = nvg_2(z - d_2u - d_2\beta v, z - d_1\alpha u - d_1v) + Mv.$$

Hence, for any $(\xi,\eta)\in I$, we have

$$F_1(\hat u,\tilde v)\leqslant F_1(\xi,\eta)\leqslant F_1(\tilde u,\hat v),\qquad F_2(\hat u,\tilde v)\geqslant F_2(\xi,\eta)\geqslant F_2(\tilde u,\hat v).$$

Then the positive property of $(-\frac{d^2}{dx^2}+M)^{-1}$ and the definition of super- and sub-solutions imply that

$$\hat u\leqslant\left(-\frac{d^2}{dx^2}+M\right)^{-1}F_1(\hat u,\tilde v)\leqslant\left(-\frac{d^2}{dx^2}+M\right)^{-1}F_1(\xi,\eta)\leqslant\left(-\frac{d^2}{dx^2}+M\right)^{-1}F_1(\tilde u,\hat v)\leqslant\tilde u,$$
$$\tilde v\geqslant\left(-\frac{d^2}{dx^2}+M\right)^{-1}F_2(\hat u,\tilde v)\geqslant\left(-\frac{d^2}{dx^2}+M\right)^{-1}F_2(\xi,\eta)\geqslant\left(-\frac{d^2}{dx^2}+M\right)^{-1}F_2(\tilde u,\hat v)\geqslant\hat v,$$

which means $FI\subset I$. The proof is finished. □

Define functions $H_2(m), H_1(n)$ by

$$H_1(n)=\inf\bigl\{m:\ m>G_1(m,n)\bigr\},\qquad H_2(m)=\inf\bigl\{n:\ n>G_2(m,n)\bigr\}. \tag{3.8}$$

It suffices to show that the boundary of Σ consists of the two curves Γ_1: $m=H_1(n)$, Γ_2: $n=H_2(m)$, which are increasing with respect to n and m, respectively. This property can be described by considering the horizontal and vertical slices of Σ which are given by

$$S_h(n)\equiv\bigl\{m>\lambda_1:\ (m,n)\in\Sigma\bigr\},\qquad S_v(m)\equiv\bigl\{n>\sigma_1:\ (m,n)\in\Sigma\bigr\}.$$

Theorem 3.3. *Let $H_2(m), H_1(n)$ be defined by* (3.8)*. Then the set Σ of parameters (m,n) for which* (EP) *has a positive solution is an unbounded region in R^2_+ whose boundary consists of two curves*

$$\Gamma_1:\ m=H_1(n)\quad\text{and}\quad\Gamma_2:\ n=H_2(m)$$

in the following sense: for each $n>\sigma_1$ and $\hat\lambda_1(n)\neq\hat\sigma_1^{-1}(n)$, the horizontal slice $S_h(n)$ of Σ is a non-empty interval whose left endpoint is $m=H_1(n)$; for each $m>\lambda_1$ and $\hat\sigma_1(m)\neq\hat\lambda_1^{-1}(m)$, the vertical slice $S_v(m)$ is a non-empty interval whose lower endpoint is $n=H_2(m)$. Here $\hat\sigma_1^{-1}(n)$ and $\hat\lambda_1^{-1}(m)$ are the inverse functions of $\hat\sigma_1(m)$ and $\hat\lambda_1(n)$, respectively.

Proof. We show that for each $n>\sigma_1$ and $\hat\lambda_1(n)\neq\hat\sigma_1^{-1}(n)$, the set $S_h(n)$ is a non-empty interval whose left endpoint is $m=H_1(n)$. The proof for the vertical slice $S_v(m)$ is similar. For $n>\sigma_1$ and $\hat\lambda_1(n)\neq\hat\sigma_1^{-1}(n)$, let

$$a\equiv a(n)=\inf\bigl\{m:\ m\in S_h(n)\bigr\},\qquad b\equiv b(n)=\sup\bigl\{m:\ m\in S_h(n)\bigr\},$$

where b may be ∞. We first show the interval (a,b) is non-empty, and each point $m\in(a,b)$ belongs to $S_h(n)$. It is easy to see that the non-emptiness of (a,b) follows directly from Theorem 2.3, which asserts that (EP) has a positive solution when m lies between $\hat\lambda_1(n)$ and $\hat\sigma_1^{-1}(n)$. Hence $a<b$. Let $m\in(a,b)$. Then there exist $m_1,m_2\in S_h(n)$ such that $m_1<m<n_1$. Then $(m_1,n),(n_1,n)\in\Sigma$. Next, we show (EP) has a positive solution at (m,n), that is, F has a positive fixed point in $\dot D$. Denote the positive solution of (EP) with $(m,n)=(m_1,n)$ and $(m,n)=(n_1,n)$ by (u_m,v_m) and (u_n,v_n), respectively, where $m_1<n_1$. Then the two pairs of $(\frac{z}{d_2}+1,v_m)$ and $(u_m,0)$, $(u_n,\frac{z}{d_1}+1)$ and $(0,v_n)$ are both ordered super- and sub-solutions of (EP). Let

$$I_1=\left\{(u,v):\ u_m\leqslant u\leqslant\frac{z}{d_2}+1,\ 0\leqslant v\leqslant v_m\right\},$$
$$I_2=\left\{(u,v):\ 0\leqslant u\leqslant u_n,\ v_n\leqslant v\leqslant\frac{z}{d_1}+1\right\},$$
$$I_0=\left\{(u,v):\ 0\leqslant u\leqslant\frac{z}{d_2}+1,\ 0\leqslant v\leqslant\frac{z}{d_1}+1\right\}.$$

It is clear that I_i $(i=0,1,2)$ are bounded, convex sets and $FI_i\subset I_i$ $(i=0,1,2)$ for sufficiently large M by Lemma 3.2. By virtue of the Schauder's fixed point theorem [1], F has at least one fixed point at each set I_0,I_1,I_2. If any one of these fixed points is positive then we are done. Moreover, it also follows from Schauder's fixed point theorem that

$$\operatorname{index}(F,I_i,I_i)=1\quad(i=0,1,2).$$

Suppose F has only trivial or semi-trivial fixed point in I_0. Then the only fixed points in I_0 are $(0,0)$, $(\theta_m,0)$ and $(0,\vartheta_n)$. Since, by assumption, (u_m,v_m) and (u_n,v_n) are positive, it follows that $(0,0)\notin I_1\cup I_2$, $(\theta_m,0)\in I_1$, $(0,\vartheta_n)\in I_2$. Let $P_\sigma(0,0)(\sigma\ll1)$ be a small neighborhood of $(0,0)$ in W. Then by Lemma 2.6, it is easy to see that $\operatorname{index}(F,P_\sigma(0,0),I_0)=0$. Since $(\theta_m,0)\in I_1$, $(0,\vartheta_n)\in I_2$, we have

$$u_m \leqslant \theta_m \leqslant \frac{z}{d_2}+1, \qquad v_n \leqslant \vartheta_n \leqslant \frac{z}{d_1}+1, \quad x \in (0,1).$$

By the strong maximum principle, it is easy to check that

$$u_m < \theta_m < \frac{z}{d_2}+1, \qquad v_n < \vartheta_n < \frac{z}{d_1}+1, \quad x \in (0,1).$$

Hence, there exist small open neighborhoods $O^+(\theta_m,0)$ and $O^+(0,\vartheta_n)$ of $(\theta_m,0)$ and $(0,\vartheta_n)$ in W, respectively, such that $O^+(\theta_m,0) \subset I_1$, $O^+(0,\vartheta_n) \subset I_2$. Moreover, the neighborhoods are open in I_0 and F has no fixed point in $I_1 \setminus O^+(\theta_m,0)$ and $I_2 \setminus O^+(0,\vartheta_n)$. By the permanence and excision property of the index,

$$\begin{aligned}
&\operatorname{index}\big(F, O^+(\theta_m,0), I_0\big) = \operatorname{index}\big(F, O^+(\theta_m,0), I_1\big) = \operatorname{index}(F, I_1, I_1) = 1,\\
&\operatorname{index}\big(F, O^+(0,\vartheta_n), I_0\big) = \operatorname{index}\big(F, O^+(0,\vartheta_n), I_2\big) = \operatorname{index}(F, I_2, I_2) = 1.
\end{aligned}$$

Thus

$$\operatorname{index}\big(F, P_\sigma(0,0), I_0\big) + \operatorname{index}\big(F, O^+(\theta_m,0), I_0\big) + \operatorname{index}\big(F, O^+(0,\vartheta_n), I_0\big) = 2,$$

which contradicts

$$\operatorname{index}\big(F, P_\sigma(0,0), I_0\big) + \operatorname{index}\big(F, O^+(\theta_m,0), I_0\big) + \operatorname{index}\big(F, O^+(0,\vartheta_n), I_0\big) = \operatorname{index}(F, I_0, I_0) = 1.$$

This means that F has a fixed point in I_0 which is different from $(0,0)$, $(\theta_m,0)$ and $(0,\vartheta_n)$.

It remains to show that the left endpoint is $a = H_1(n)$. For any $m \in (a,b)$ and $m \neq \hat{\lambda}_1(n)$, Theorem 3.1 implies that $m > G_1(m,n)$, which ensures $m > H_1(n)$ by the definition of $H_1(n)$, and consequently $a \geqslant H_1(n)$. On the other hand, for any $m < a$, one can assert that $(m,n) \notin \Sigma$. It follows from Theorem 2.3 that $m \neq \hat{\lambda}_1(n)$, $n \neq \hat{\sigma}_1(m)$. In view of Theorem 3.1, either $m < G_1(m,n)$ or $n < G_2(m,n)$. Now for $m < a \leqslant \hat{\lambda}_1(n)$, the solution $\underline{u}_0$ of (3.1) is 0, therefore the solution $\underline{u}(x,t)$ of (3.5) with $u(x,0) = \underline{u}_0$ is identically 0. Thus $\underline{u}_s \equiv 0$. By the definition of $G_2(m,n)$, it is easy to see that $G_2(m,n) = \lambda_1(g_2(z,z)) = \sigma_1 < n$, which leads to $m < G_1(m,n)$. In view of the arbitrariness of m, we must have that $a \leqslant H_1(n)$. This leads to the relation $a = H_1(n)$, which completes the proof of this theorem. □

Theorem 3.4. *Suppose* (m,n) *satisfies*

(i) $H_1(n) < m < \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$;
(ii) $H_2(m) < n < \hat{\sigma}_1(m)$, $m > \hat{\lambda}_1(n)$.

Then (EP) *has at least two positive solutions.*

Proof. We only prove the case (i), the proof for the case (ii) is similar. Let c be any constant satisfying $H_1(n) < c < m$, and let (u_c, v_c) be a positive solution of (EP) with $(m,n) = (c,n)$. The existence of (u_c, v_c) follows from Theorem 3.1. Then it is easy to see that the pair $(\frac{z}{d_2}+1, v_c)$, $(u_c, 0)$ is ordered super- and sub-solutions of (EP). Set

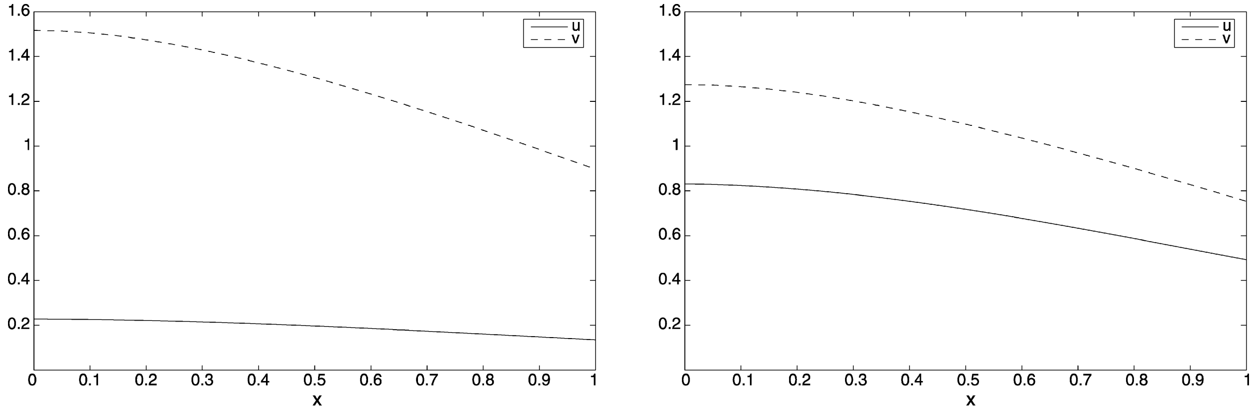
$$\begin{aligned}
I_0 &= \left\{(u,v) \in C\big([0,1], R^2\big):\ 0 \leqslant u \leqslant \frac{z}{d_2}+1,\ 0 \leqslant v \leqslant \frac{z}{d_1}+1\right\},\\
\tilde{I}_1 &= \left\{(u,v) \in C\big([0,1], R^2\big):\ u_c \leqslant u \leqslant \frac{z}{d_2}+1,\ 0 \leqslant v \leqslant v_c\right\}.
\end{aligned}$$

Then it follows from Lemma 3.2 that $FI_0 \subset I_0$, $F\tilde{I}_1 \subset \tilde{I}_1$ for sufficiently large M. By Schauder's fixed point theorem, we obtain that

$$\operatorname{index}(F, I_0, I_0) = 1, \qquad \operatorname{index}(F, \tilde{I}_1, \tilde{I}_1) = 1.$$

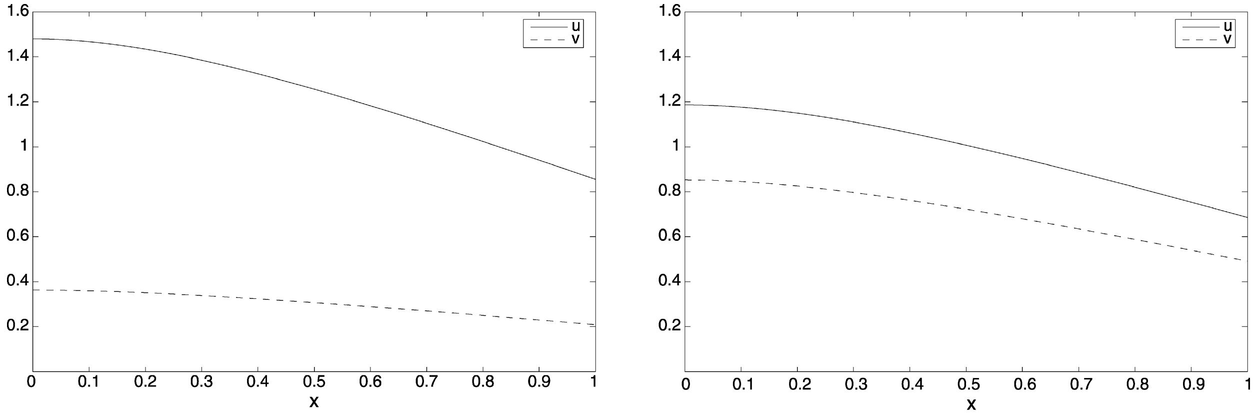
Next, we show that F has a positive fixed point in $\tilde{I}_1$ and a positive fixed point in $I_0 \setminus \tilde{I}_1$. First, suppose F has no positive fixed point in $\tilde{I}_1$. Clearly, $(\theta_m, 0) \in \tilde{I}_1$ and there exists a small neighborhood $O^+(\theta_m,0) \subset \tilde{I}_1$, and $O^+(\theta_m,0)$ is open in $\tilde{I}_1$. Thus the relation $n > \hat{\sigma}_1(m)$ implies $\operatorname{index}(F, O^+(\theta_m,0), I_0) = \operatorname{index}(F, O^+(\theta_m,0), \tilde{I}_1) = 0$. Using $\operatorname{index}(F, \tilde{I}_1, \tilde{I}_1) = 1$ and $(0,0), (0,\vartheta_n) \notin \tilde{I}_1$, we know that F has at least one positive fixed point in $\tilde{I}_1$. The remainder task is to show F has a positive fixed point in $I_0 \setminus \tilde{I}_1$. Suppose F has no positive fixed point in $I_0 \setminus \tilde{I}_1$. For each fixed point $U_i = (u_i, v_i)$, $i = 1, 2, \ldots, N$ of F in $\tilde{I}_1$, there is a small open neighborhood $B_i \subset \tilde{I}_1$ of I_0. Let $B = \bigcup B_i$. Then B contains all the fixed points of F in $\tilde{I}_1$ and F has no fixed point on ∂B. Thus by the permanence and excision property of the index, we have

$$\operatorname{index}(F, B, I_0) = \operatorname{index}(F, B, \tilde{I}_1) = \operatorname{index}(F, \tilde{I}_1, \tilde{I}_1) = 1.$$



(a) (b)

Fig. 1. Two positive solutions of (EP) with $m = 3$, $n = 4.84$. In this case, we checked that $m < \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$ by numerical calculation method, which is given by [20].



(a) (b)

Fig. 2. Two positive solutions of (EP) with $m = 4$, $n = 4.16$. Moreover, we also checked that $m > \hat{\lambda}_1(n)$, $n < \hat{\sigma}_1(m)$ by numerical calculation method in [20].

On the other hand, the assumption $H_1(n) < m < \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$ implies that $\operatorname{index}(F, P_\sigma(0,0), I_0) = 0$. Repeating the same arguments as in Lemma 2.7, it is easy to see that $\operatorname{index}(F, O^+(0,\vartheta_n), I_0) = 1$. Hence

$$\operatorname{index}(F, I_0, I_0) = \operatorname{index}(F, B, I_0) + \operatorname{index}\big(F, P_\sigma(0,0), I_0\big) + \operatorname{index}\big(F, O^+(0,\vartheta_n), I_0\big) = 2,$$

which contradicts $\operatorname{index}(F, I_0, I_0) = 1$. This shows that F has at least a positive fixed point in $I_0 \setminus \tilde{I}_1$. That is, (EP) has at least two positive solutions. $\square$

Remark. It follows from Theorem 3.4 that the multiple positive steady states of (EP) exist when $H_1(n) < m < \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$ or $H_2(m) < n < \hat{\sigma}_1(m)$, $m > \hat{\lambda}_1(n)$. For example, take the parameters as follows: $d_1 = 1$, $d_2 = 1$, $\alpha = 0.4$, $\beta = 0.6$, $\gamma = 1$, $m_{s_1} = 2$, $m_{s_2} = 2.75$, $m_{r_1} = 2.5$, $m_{r_2} = 2$, $k_{s_1} = 1$, $k_{s_2} = 1.5$, $k_{r_1} = 1.5$, $k_{r_2} = 2$, and $m = 3$, $n = 4.84$ in Fig. 1 and $m = 4$, $n = 4.16$ in Fig. 2. The numerical computations indicate that the multiple positive steady-state solutions of (EP) do exist in the two cases above (see Figs. 1 and 2). Moreover, by numerical calculation method, which is given by [20], we checked that $m < \hat{\lambda}_1(n)$, $n > \hat{\sigma}_1(m)$ when $m = 3$, $n = 4.84$; and $m > \hat{\lambda}_1(n)$, $n < \hat{\sigma}_1(m)$ when $m = 4$, $n = 4.16$.

Acknowledgments

The authors would like to give their sincere thanks to the anonymous referees for their valuable suggestions leading to an improvement of the paper.

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